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Nonlinear Quantum Mechanics is a Classical Theory

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Quantum mechanics with nonlinear operators is shown to be an essentially classical theory. A general scheme of delinearization of a quantum theory is described.

1. INTRODUCTION

For many years there have been attempts to incorporate nonlinear operators into the standard formalism of quantum mechanics (see, e.g., Bohm, 1987; Mielnik, 1974; Białynicki-Birula and Mycielski, 1976; Davies, 1979; Gisin, 1981; Weinberg, 1989). At first sight the acceptance of nonlinear operators seems to cause only minor changes of the theory itself; nonlinearities are expected to help in removing some quantal "paradoxes" or, as in Weinberg's (1989) proposal, to test the standard quantum mechanics. It should be stressed, however, that nonlinear quantum mechanics, i.e., the standard quantum mechanics supplemented by nonlinear operators as representing observables, is not merely an "improved" or "generalized" quantum mechanics, but a totally reconstructed new theory. In fact, a nonlinear quantum mechanics, if such a theory is ever to become a mature one, has to be a classical (nonquantal) theory of quantal phenomena. We want to demonstrate here the dramatic collapse of the elaborated structure of quantum mechanics caused by introducing nonlinear observables.

It is rather obvious that such a demonstration demands a theoretical framework which would be sufficiently general to cover quantal as well as classical theories. It appears that the so-called "operational approach" to statistical theories (Davies, 1976; Ludwig, 1983) suits well this purpose. Our considerations are based on known mathematical results which are found in Alfsen (1971) and Asimov and Ellis (1980). It also should be acknowledged that some of the basic ideas of this paper have been known

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for years. Thus, the description of quantal mixed states as probability measures on pure states was proposed by Misra (1974) and Ghirardi *et al.* (1976), the phenomenon of distinguishing statistical mixtures of quantal pure states by nonlinear operators was demonstrated by Haag and Bannier (1978), and the transformations of a physical theory caused by a restriction or an extension of the set of admissible measurements were observed by Holevo (1982) and Neumann (1985). The general conclusion of this paper, that nonlinear quantum mechanics is inevitably a classical theory, was announced as early as 10 years ago (Bugajski, 1979, 1980, 1981); however, it seems even more important now in the face of growing interest in nonlinear modifications of quantum mechanics.

Before we come to the point, we should introduce several notions. Let V be an order-unit Banach space. Elements of the order interval $[o, e] \subset V$, where o is the origin and e is the order unit of V, are called effects. The set of effects is endowed with the order inherited from V and two partial operations "+" and "−" defined as the restrictions to $[o, e]$ of the corresponding linear operations over V. The set of effects is always convex; its set of extreme effects will be denoted $Ex[o, e]$. The extreme elements of $[o, e]$ are called sharp effects (decision effects in terms of the Marburg school of Ludwig), whereas the other effects are fuzzy. An observable (related to V) is a vector-valued measure $A: \mathbb{B}(\mathbb{R}) \mapsto [o, e]$, where $\mathbb{B}(\mathbb{R})$ is the σ-field of Borel subsets of the real line $\mathbb{R}$, with $A(\mathbb{R}) = e$. If the range of an observable A is contained in $Ex[o, e]$, we call it sharp; otherwise, it is unsharp (fuzzy if the range of A consists of fuzzy effects only). Two effects are called compatible (comeasurable) if they both belong to the range of an observable; two observables are compatible if the set join of their ranges consists of pairwise compatible effects only.

The Banach dual of (an order-unit Banach space) V is a base-norm Banach space V^*, the base S of V^* is called the set of states (of V). S is convex and weak-$*$ compact, its extreme elements are called pure states. We will say that $W \subset V$ separates elements of $T \subset S$ iff for any $\alpha, \beta \in T$ there is $a \in W$ such that $\alpha(a) \neq \beta(a)$. The interval $[o, e]$ always separates all states of S. The real number $\alpha(a)$, $\alpha \in S$, $a \in [o, e]$ is called the probability (of occurrence) of the effect a in the state α. A composition of a state α and an observable A defines a probability measure on $\mathbb{R}$.

S will be called a simplex (more precisely, a Bauer simplex) iff for every point α of S there exists a unique probability measure μ supported by $\overline{ExS}$ (the weak-$*$ closure) such that

$$\alpha(a) = \int_{\overline{ExS}} \beta(a)\, d\mu(\beta)$$

for any $a \in [o, e] \subset V$. If S is a simplex, then $ExS = \overline{ExS}$, S can be identified with $M(\overline{ExS})_1^+$— the set of probability measures on $\overline{ExS}$, and V can be identified with $C(\overline{ExS})$, the order-unit Banach space of all continuous real functions on the compact Hausdorff space $\overline{ExS}$. If (and only if) S is a simplex, V is a lattice in its natural ordering and $[o, e]$ is a lattice. If S is a simplex, then V will be called classical. Otherwise, we will say that V is quantal.

2. QUANTUM MECHANICS WITH NONLINEAR OPERATORS

The abstract notions of Section 1 are well illustrated by the standard quantum mechanics, i.e., the handbook theory based on a separable Hilbert space $\mathscr{H}$ without superselection rules. The set $\mathscr{L}_s(\mathscr{H})$ of all self-adjoint bounded (linear) operators on $\mathscr{H}$ corresponds to V with its natural structure of an order-unit Banach space with respect to the operator norm. The sharp observables related to $\mathscr{L}_s(\mathscr{H})$ are then projection-valued measures which, considered as spectral ones, define self-adjoint operators on $\mathscr{H}$. This time-honored representation of quantum sharp observables by self-adjoint operators is hardly possible for unsharp ones.

Instead of S as defined in Section 2, the standard quantum mechanics considers a weak-* dense subset S_n of S consisting of all ultraweakly continuous real linear functionals on $\mathscr{L}_s(\mathscr{H})$. S_n, the set of normal states, is identified with the base of the base-normed Banach space $\mathscr{T}_s(\mathscr{H})$ of all self-adjoint operators of the trace class on $\mathscr{H}$. $\mathscr{T}_s(\mathscr{H})$ is the predual of $\mathscr{L}_s(\mathscr{H})$.

Every observable defines the mean-value function on ExS_n, which for a bounded observable can be extended to a continuous real linear functional on $\mathscr{T}_s(\mathscr{H})$. Thus, to every (bounded) observable is attached, via the mean-value function on ExS_n, an element of $\mathscr{L}_s(\mathscr{H})$. This correspondence between observables and operators on $\mathscr{H}$ is one-to-one if we restrict ourselves to the sharp bounded observables and then agrees with the spectral theorem. Any mean-value function on ExS_n also can be uniquely extended over $\overline{ExS}$ and hence over $\mathscr{L}_s(\mathscr{H})^*$, defining then a weak-* continuous linear functional on $\mathscr{L}_s(\mathscr{H})^*$.

Nonlinear "observables" are usually introduced in the standard quantum mechanics in the traditional way as nonlinear operators acting on $\mathscr{H}$. There is nothing like a spectral theorem for nonlinear operators, hence they cannot be considered as observables in the sense defined above. The recent, and up to now the best developed, proposal of Weinberg (1989), where nonlinear "observables" are defined as nonbilinear functions on $\mathscr{H} \times \mathscr{H}$, also suffers this shortcoming. Nevertheless, any nonlinear operator (as well as any Weinberg nonbilinear function) defines a function M_B on $\mathscr{H}$ formally in the same way as any sharp observable defines the mean-value function on pure states, $M_B(\psi) := (\psi, B\psi)$ for a nonlinear operator B and $\psi \in \mathscr{H}$. The function M_B is considered as the "mean value" (or the "expectation value") of a nonlinear "observable" (represented by B) in the pure state ψ.

We should distinguish two fundamental classes of nonlinear generalizations of quantum mechanics: (i) homogeneous, where any nonlinear "observable" represented by a nonlinear operator B has to satisfy the homogeneity condition $B(c\psi) = cB(\psi)$ for any complex number c and any $\psi \in \mathcal{H}$, and (ii) nonhomogeneous, where the homogeneity condition does not need to be satisfied. We will restrict our considerations to the first case. Nonhomogeneous "observable" should distinguish different normalizations of ψ, so such an "observable" should be related—according to the ideas of the "operational approach"—with a measuring device which gives different expectation values for statistical ensembles with different numbers of samples of the investigated physical system. This means that the "expectation value" of such a nonhomogeneous "observable" could not be even measured in the standard sense, as any attempt to repeat the measurement or to increase its accuracy by increase of the number of samples would disturb the "expectation value" itself. Thus, nonhomogeneous "observables" could be accepted only if we change the traditional interpretation of $\|\psi\|$, which cannot be done in the frame of the "operational approach." There are also other reasons to reject the nonhomogeneous case—see Haag and Bannier (1978) and Weinberg (1989).

Taking into account the homogeneity condition, we conclude that the "mean-value" functions for nonlinear "observables" are defined on ExS_n. We will assume that they are defined on the whole ExS_n, and are bounded and continuous with respect to the topology induced on ExS_n by the weak Banach one. Obviously any bounded observable (i.e., any bounded effect-valued measure on $\mathbb{R}$) also defines a similar function. However, M_A for A—an observable can be always linearly extended over S_n, which is hardly ever possible for a nonlinear "observable." Let us assume that M_B (defined on ExS_n) for some nonlinear operator B can be extended on S_n. Then it also can be extended to a continuous linear functional on $\mathcal{T}_s(\mathcal{H})$, which in turn is an element of $\mathcal{L}_s(\mathcal{H})$. Thus, $M_B = M_A$ for some sharp observable A. We conclude that if only M_B does not coincide with M_A for some observable A, then M_B has no linear extension over S_n. This provides a formal proof of the known fact that it is impossible to find a consistent definition of the expectation value for a nonlinear "observable" in quantal mixed states. Sometimes this property is referred to as the impossibility of defining the trace for nonlinear operators.

One could hope to overcome this obstacle by extending linearly M_B (for a nonlinear "observable" B) over statistical mixtures of pure states. Here we touch upon the specific quantal feature called "nonunique decomposability of quantum mixtures" (see, e.g., Beltrametti, 1985). To make the discussion simpler, we will assume that nonlinear "observables" are represented by mean-value functions on $\overline{ExS}$, so by elements of the order-unit Banach space $C(\overline{ExS})$. The mean-value functions of observables form then a norm-closed linear subspace, denoted $\mathscr{L}$, of $C(\overline{ExS})$. If we want to extend functions belonging to $C(\overline{ExS})$ over formal convex combinations of elements of $\overline{ExS}$, we arrive in a natural way at $M(\overline{ExS})_1^+$—the set of all probability measures on $\overline{ExS}$, which from the physical point of view is the set of all statistical mixtures of quantal pure states. $M(\overline{ExS})_1^+$ is the base of the base-normed Banach space $M(\overline{ExS})$ of all Baire (regular Borel) signed measures on $\overline{ExS}$. As $M(\overline{ExS}) = C(\overline{ExS})^*$ (the Banach dual), we conclude that any function of $C(\overline{ExS})$ can be linearly extended over $M(\overline{ExS})_1^+$ and that $C(\overline{ExS})$ separates $M(\overline{ExS})_1^+$. Translated into a less formal language, it means that we can consistently define "expectation values" of nonlinear "observables" in statistical mixtures of quantal pure states and that nonlinear "observables" distinguish all such statistical mixtures.

It is nearly obvious that $\mathscr{L} \subset C(\overline{ExS})$ does not separate elements of $M(\overline{ExS})_1^+$. Let us assume the opposite. Then the annihilator of $\mathscr{L}$, $\mathscr{L}^\perp := \{\alpha \in M(\overline{ExS}) | \alpha(M_A) = 0 \text{ for all } M_A \in \mathscr{L}\}$, would contain only the zero measure [the origin of $M(\overline{ExS})$]. It is known, however, that $M(\overline{ExS})/\mathscr{L}^\perp$ is isometric to $\mathscr{L}_s(\mathscr{H})^*$. Then the triviality of $\mathscr{L}^\perp$ implies that $M(\overline{ExS})$ itself should be isometric to $\mathscr{L}_s(\mathscr{H})^*$. This is a contradiction, because the base of $M(\overline{ExS})$ is a simplex, which surely does not hold for $\mathscr{L}_s(\mathscr{H})^*$. Thus, $\mathscr{L}^\perp$ must be nontrivial, which leads to the mentioned "nonunique decomposability of quantum mixtures." Indeed, the projection of $M(\overline{ExS})$ onto the quotient space $M(\overline{ExS})/\mathscr{L}^\perp$ is many-to-one when $\mathscr{L}^\perp \neq \{0\}$. As the quotient space is isometric to $\mathscr{L}_s(\mathscr{H})^*$, we see that many different probability measures on $\overline{ExS}$ are identified by $\mathscr{L}_s(\mathscr{H})$ with one nonpure quantal state.

The idea of expanding S (or S_n) to $M(ExS_n)_1^+$ appeared many years ago as a way of resolving the annoying paradoxes of the quantal description of composed systems (Misra, 1974; Ghirardi *et al.*, 1976). However, as it was shown above, quantal observables do not separate elements of $M(ExS_n)_1^+$; hence, if we want safely to introduce statistical mixtures of pure states in place of the quantal nonpure states we should look for a larger class of observables than the quantal (sharp and unsharp) ones. The first suggestion that nonlinear "observables" could do that job we owe to Haag and Bannier (1978).

The collapse of $M(\overline{ExS})_1^+$ to a nonsimplectic convex set $\mathscr{L}_s(\mathscr{H})_1^{*+}$ as a result of restricting $C(\overline{ExS})$ to its subspace $\mathscr{L}$ is well known in modern functional analysis. It was also observed by several physicists (Bugajski, 1980, 1981; Holevo, 1982; Neumann, 1985; Ludwig, 1990), who considered it merely as a curiosity. In the context of nonlinear generalizations of quantum mechanics it acquires importance as an indication of the profound difference between the standard quantum mechanics and its nonlinear generalizations. We have demonstrated above that the introduction of nonlinear "observables" (even in so lame a way as is usually done) forces us to leave the standard theory based on $\mathscr{L}_s(\mathscr{H})$ and S for a new one based on $C(\overline{ExS})$ and $M(\overline{ExS})_1^+$. The latter is, however, no longer quantal, but rather essentially classical, at least in the formal meaning.

The striking disclosure of the classical character of nonlinear quantum mechanics is a result of applying to it the ideas of the operational approach. Nevertheless, it is evident that what we have done above is by no means a satisfactory formulation of nonlinear quantum mechanics as a statistical theory. The notion of nonlinear "observable" does not satisfy the definition of observable; its fundamental shortcoming is that it does not provide rules for calculating the probability distribution of results of a measurement of a nonlinear "observable." Another weak point of our considerations is the identification of nonlinear "observables" as elements of $C(ExS)$. From a more general point of view it is a rather unfounded extrapolation, because the tight connection between observables and elements of the basic order-unit Banach space is an accidental feature of the standard quantum mechanics. Thus, we see that the procedure of "delinearization" of the standard quantum mechanics needs a precise formulation. We will do this below, having in mind the basic conclusions of the above considerations.

3. DELINEARIZATION OF A QUANTUM THEORY

If we want to formulate a correct (from the "operational" point of view) scheme of delinearization of a quantum theory we should avoid misleading suggestions coming from the specific features of the standard quantum mechanics. This is the reason that we start from a general order-unit Banach space V_q which houses the set $[o, e]_q$ of all effects of the quantum theory in question. We assume that the set S_q of states of V_q is not a simplex, to assure the minimum (perhaps also the essence) of the quantal character of the theory based on V_q and S_q.

The space V_q admits now a functional representation $D\colon V_q \mapsto C(\Omega)$, where $C(\Omega)$ is the order-unit Banach space of continuous functions on $\Omega := \overline{ExS_q}$. D is a linear endomorphism; we obtain it as a restriction to Ω of the canonical representation of V_q as the space of those weak-* continuous affine functions on S_q which can be extended over V_q^*. According to the Kadison theorem, the map D provides the smallest separating functional representation of V_q. The image of V_q under D, which will be identified with V_q itself, is a norm closed proper subspace of $C(\Omega)$. It is evident that D puts $[o, e]_q$ into $[o, e]_c \subset C(\Omega)$; the extreme effects of V_q (if there are any nontrivial ones) are in general mapped on fuzzy effects of $C(\Omega)$.

As a consequence of the considerations of Section 2, we assume that $C(\Omega)$ is the basic constituent of the delinearized theory. We are not going, however, to consider elements of $C(\Omega)$ as representing nonlinear "observables." Observables of the delinearized theory have to be defined as $[o, e]_c$-valued measures on $\mathbb{R}$. Thus, $[o, e]_c$ will get a physical interpretation as the set of all effects (elementary observables) of the delinearized theory. All linear combinations of elements of $[o, e]_c$ span the space $C(\Omega)$, so the statistical mixtures represented by elements of $M(\Omega)_1^+$ are separated by $[o, e]_c$. This is a general formulation of the statement: "nonlinear observables distinguish statistical mixtures of quantum pure states."

On the other hand, it is also clear that V_q does not separate elements of $M(\Omega)_1^+$, as the proof presented in Section 2 admits a straightforward generalization. Thus, we see that even in this general formulation we can state that "quantal observables do not distinguish statistical mixtures of quantal pure states." The quotient map Q: $M(\Omega) \mapsto V_q^*$ is then nontrivial, so the integral representation of points of S_q is nonunique. For a given $\alpha \in S_q$ any probability measure μ belonging to $Q^{-1}(\alpha)$ is a representing measure for α, i.e., satisfies the relation $\alpha(a) = \int_\Omega \beta(a)\, d\mu(\beta)$ for any $a \in [o, e]_q$. Observe that $\beta(a)$ for fixed $a \in [o, e]_q$ and variable $\beta \in \Omega$ is simply the image of a under the delinearization map D. The nonunique representation of $\alpha \in S_q$ by a probability measure on Ω (for $\alpha \in \Omega \subset S_q$ the representation is unique) is an abstract formulation of the "nonunique decomposability of quantum mixtures."

It should be observed that $[o, e]_c$ does not have in general extreme elements except o and e. There is, however, a well-established tradition that the set of effects should possess a rich collection of extreme points, including in the classical case the Boolean algebra of measurable sets in question. A natural way to satisfy this demand would be to pass to the second dual $C(\Omega)^{**}$ in order to obtain the weak-* compactification of $[o, e]_c$, unfortunately, it provides us with too large set of extreme effects. As a compromise we propose to consider, instead of $C(\Omega)$, the order-unit Banach space of Baire functions on Ω; we will denote it V_{cl}. The set $Ex[o, e]_{\mathrm{cl}}$ of extreme effects of V_{cl} consists exactly of all Baire subsets of Ω and is a Boolean lattice under the inherited order. V_{cl} is a weak-* dense subspace of $C(\Omega)^{**}$; we will consider it together with the induced topologies. The original space $C(\Omega)$ is naturally identified with a norm closed subspace of V_{cl}.

The delinearization map D can be now considered as a linear endomorphism of V_q into V_{cl}; hence, it preserves (among others) the order of V_q. This implies that D transforms the quantal effects $[o, e]_q$ into the classical ones: $D([o, e]_q) \subset [o, e]_{cl}$. D restricted to $[o, e]_q$ has the following properties: $D(o) = o$, $D(e) = e$, $D(a+b) = D(a) + D(b)$ if $a+b$ does exist in $[o, e]_q$, $D(a-b) = D(a) - D(b)$ if $a-b$ does exist in $[o, e]_q$. These properties together with the norm-closeness of V_q in V_{cl} suffice for a safe transportation of original quantal observables into the delinearized theory. All observables related to V_q are then observables of the delinearized theory, i.e., if $A: \mathbb{B}(\mathbb{R}) \mapsto [o, e]_q$ is an observable, then its composition with the delinearization map $D \circ A: \mathbb{B}(\mathbb{R}) \mapsto [o, e]_{cl}$ is an observable as well. This does not mean, of course, that D preserves also the "name," i.e., the physical interpretation of an observable: it could happen that, e.g., the energy observable of the original quantum theory is not the energy observable of the delinearized theory.

The preservation of observables by D provides a safe basis for the opinion that the delinearized theory is an extension of the original one. Nevertheless, we should be aware of quite dramatic changes caused by the delinearization. The notorious "quantum logic" $Ex[o, e]_q$ is mapped by D on nonextreme (fuzzy) effects of $[o, e]_{cl}$ and loses almost all its nice properties (assuming that for a given V_q it makes sense to talk about a "quantum logic" at all). We should accept it without worry, as even in the operational quantum mechanics, where the legitimate successor of the "quantum logic" $Ex[o, e]_q$ is the whole set of effects $[o, e]_q$, the majority of standard properties of "quantum logic" is not valid. On the other hand, the destruction of "quantum logic" by D is a necessary condition for the realization of the delinearization procedure. Namely, the map D materializes the idea of "hidden variables" for the original quantum theory. In spite of all celebrated "no-go" theorems, it appears fairly possible just because D transfers the original quantum theory into a classical operational theory and maps (eventual) sharp effects and sharp observables onto fuzzy ones.

It also should be stressed that the relations among quantal observables are not in general preserved by D. The theory resulting from the delinearization is entirely classical, so all its observables, including these transformed from the quantum theory, are comeasurable. Thus, the delinearization invalidates commutativity, complementarity, uncertainty, and other so widely discussed peculiarities of the set of quantal observables. For any two observables transported by D from the original quantal theory there is in the delinearized theory a join observable, which completely trivializes the question of quantum joint probabilities and related problems.

A similar situation arises for quantal pure states. The set $\Omega = \overline{ExS_q}$ occupies more or less the same position in the original quantum theory as in its delinearization. Nevertheless, the characteristic quantal features of ExS_q disappear. The most fundamental structural property of ExS_q is the existence of nontrivial transition probabilities between quantal pure states (which implies, e.g., that quantal pure states do not need to be pairwise orthogonal). In the standard quantum mechanics the transition probability is defined via the Hilbert-space inner product; it makes the impression that the transition probability is an inherent property of the set of pure states. However, the operational approach suggests that the transition probability should be related to the set of admissible operations, then to the sets of effects and observables (see, e.g., Bugajski and Lahti, 1980). Without entering into a detailed discussion here, we could propose the following definition of the transition probability $p_q(\alpha, \beta)$ for $\alpha, \beta \in \overline{ExS}_q$ in a general case: $p_q(\alpha, \beta) := 1 - \sup\{|\alpha(a) - \beta(a)| \,|\, a \in [o, e]_q\}$. It is easy to check that in the standard quantum mechanics this definition coincides (for α, β-normal pure states) with the traditional one, whereas in the general case $1 - p_q(\alpha, \beta)$ is just the statistical distance (see, e.g., Busch, 1987) restricted to $\overline{ExS}_q$. Our definition of $p_q(\alpha, \beta)$ stresses the crucial dependence of the transition probability on the "theoretical environment" of $\overline{ExS}_q$. Indeed, if we consider now $\overline{ExS}_q$ as the "phase space" of the delinearized theory, we find that any two different pure states are orthogonal (i.e., $p_{cl}(\alpha, \beta) = 0$ for any $\alpha, \beta \in \Omega$, $\alpha \neq \beta$), which is a result of extending the set of admissible effects.

4. INTERPRETATIONS

The procedure of delinearization of a quantum theory is merely a formal scheme which should be provided with an appropriate interpretation. We will briefly discuss three possible interpretations, starting from (presumably) the most conservative one.

4.1. Quantum Mechanics Is Valid

The preliminary state of development of nonlinear quantum mechanics motivates to some extent a sceptical attitude toward the above considerations. Even so, the formal scheme of delinearization could be interesting as a new technical framework.

(a_1) The delinearization provides a classical description of a quantum theory, which after some elaboration could take the place of the Wigner-Moyal one. It opens the possibility of extending over quantum theories the classical notions of statistics, information theory, etc., and provides a language in which one can compare quantum notions and their classical prototypes.

(a_2) We could see the delinearization scheme as a method of producing "quantal theories" from a classical one. Then we can use the quantization map Q to create abstract models of "quantum" theories by different choices of $V_q \subset C(\Omega)$. Such a models, even if of no practical use, could have a theoretical value as examples of non-Hilbertian "quantal" theories.

4.2. Quantum Mechanics Admits Hidden Variables

One can get the impression that the delinearization scheme provides a successful realization of the idea of "hidden variables." If this were to be so, we should see the delinearized theory as a classical theory of a hypothetical subquantum level, whereas the original quantum theory—valid on its own level of description—would be related to it as, say, classical thermodynamics is to the classical statistical mechanics. There are some arguments against this interpretation. The set of pure states (the phase space) of the delinearized theory is essentially the same as the set of pure states of the original quantum theory. This seems to contradict the idea of hidden variables, which aims to explain the quantal probabilities as resulting from a classical statistics—hence the quantal pure states should be represented by nontrivial probability measures on a hidden-variables phase space. Moreover, the delinearized theory is not exactly a classical statistical theory as it should be according to the hidden-variables idea, but rather the operational extension of a classical statistical theory admitting fuzzy (unsharp) observables.

On the other hand, the delinearization provides indeed an embedding of a quantum theory into a classical one in spite of the known negative results. This apparent contradiction is explained by the observation that the disproved version of the hidden-variables hypothesis assumes an embedding of the extreme quantum effects into the set of extreme classical effects, which does not hold here.

4.3. Quantum Mechanics Is Not Valid

It seems that the most natural interpretation of the delinearization should start with an obvious realization that the delinearized theory is a nonlinear extension of the original one. This attitude leads, however, to a radical conclusion. The embedding of V_q into V_{cl} means now that almost all specific features of quantal theories can be explained in a surprising and trivial way: they would result from restricting the set of observables of a classical theory. All the mysteries, puzzles, and paradoxes of quantum mechanics together with its allegedly inherent probabilistic character would be merely by-products of the unsufficiently motivated linearity assumption. The delinearization should then imply an acceptance (or an anticipation) of the existence of experimental capabilities ignored up to now which are to be represented by nonlinear observables. If it proves true, then quantum mechanics becomes a classical theory.

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